Variational formulation of non-equilibrium traffic models: theory and implications

Jia Li, Michael Zhang

University of California, Davis

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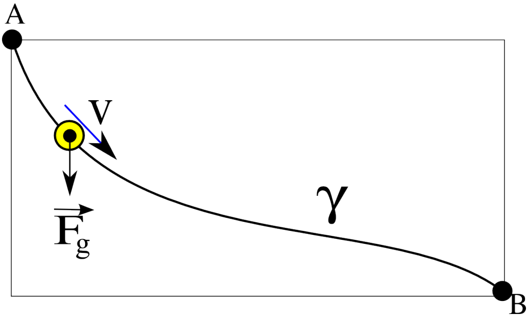
Outline

- Background
  - What is variational formulation (VF)
  - Motivation and scope of the study
- Method
  - Review of VF in scalar case
  - Second-order models
  - VF for a non-equilibrium system w/o relaxation
  - Effect of relaxation
- Implication
- Remarks

Example 1: Curve of steepest descent

Find a path connecting A and B, such that travel time is minimized (Bernoulli, 1696)

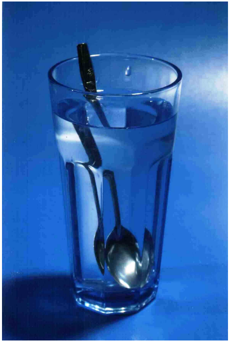


(Figure source: google)

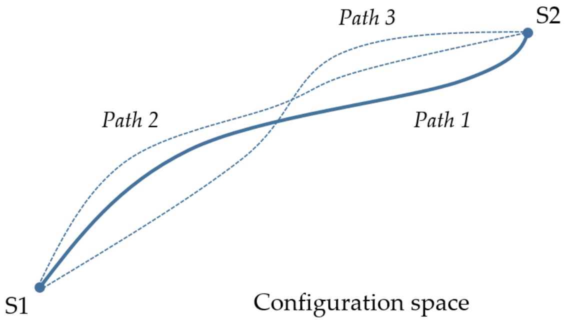
Example 2: Refraction of light – Fermat's principle

Fermat's principle (principle of least time) regarding light propagation in medium:

- Version 1: Path taken between two points by a ray of light is the path corresponding to the least travel time
- Version 2: Rays of light traverse the path of stationary optical length with respect to variations of the path



Example 3: Hamiltonian's principle



Dynamics of a physical system is determined by a variational problem for a functional based on a single function, the Lagrangian, which contains all physical information concerning the system and the forces acting on it.

Example 4: LWR model

Lighthill-Whitham-Richards (LWR, 1955, 1956) model is the simplest one that captures queue onset, growth and decay.

$$\partial_t \rho + \partial_x Q_e(\rho) = 0 \tag{1}$$

Many reasons to use it:

1. Simple to calibrate (Del Castillo, 1995)
2. Simple to implement (Daganzo, 1995; Lebacque, 1996)
3. Tractable (LeVeque, 1992)
4. Modeling flexibility (multilane, multiclass, hybrid, etc.)

Example 4: LWR model (cont'd)

The "minimum principle" (aka. variational formulation) for the LWR model

$$N_P = \min_{\gamma_{PQ}} \{N_Q + R(\gamma_{PQ})\}$$

where N represents cumulative count of traffic.

- Newell (1994) conjectured a minimum principle for LWR model with triangular fundamental diagram
- Daganzo (2005a,b) generalized and formalized this principle
- In mathematics literature, known as Lax-Hopf formula

Research question

We ask: If the variational principle is a general principle underpinning traffic dynamics? What bridges a conservation law with the variational principle?

Answers to these questions do matter, since VF presumably could

- Improve modeling flexibility
- Enhance analysis, estimation and numerical treatment
- Connect different models
- Satisfy curiosity
- ...

Milestones

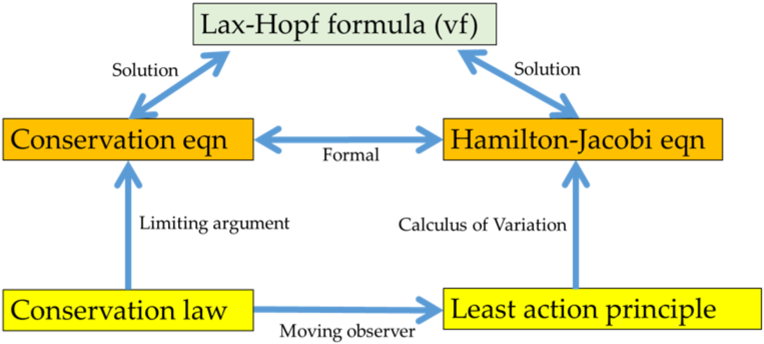
Different angles to undertand and establish VF

- Green's theorem: Lax (1957)
- Intuition: Newell (1994)
- Moving observer: Daganzo (2005a,b; 2006)
- Viability theory: Bayen and his group (from 2005)
- Lagrangian coordinate: Leclercq, Laval and Chevallier (2007); Leclercq and Laval (2013)

Applications

- Simulation
- Estimation
- Analysis

Overview


Lax-Hopf formula (vf)
Solution
Solution
Conservation eqn
Formal
Hamilton-Jacobi eqn
Limiting argument
Calculus of Variation
Conservation law
Moving observer
Least action principle

Scalar case: wave properties

Variational formulation is dictated by properties of waves (characteristics). Recall that characteristic curve of the LWR model is defined as follows

$$\dot{x} = Q_e'(\rho) \tag{2}$$

Two key observations:

- Newell (1994): Along $\dot{x} = v_f$ and $\dot{x} = -w$, dN/dt is constant. This is generalizable to the case of generic concave flux.
- Daganzo (2005a): If P and Q are connected by a wave, then the wave minimizes some 'cost functional' among all directed paths connecting them.

Scalar case: cost functional

The cost functional for a path $\gamma := x(t)$ $(t_P \leq t \leq t_Q)$ is defined as

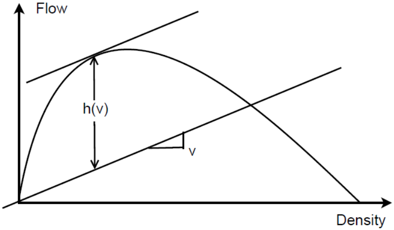
$$R(\gamma) = \int_{t_P}^{t_Q} (Q_e(\rho) - \rho \dot{x}) dt$$

Note:

- R is Legendre-Fenchel transform of fundamental diagram Q_e
- R and Q_e are said dual to each other in convex optimization context; they represent different ways of expressing the same relation
- R is nothing else than total passing flow relative to a moving observer with trajectory x

Scalar case: a key result

If the a moving observer starts from boundary $\mathcal{B}$ and reach Q in the end, what is the total passing flow relative to him?



Key result: Along any path γ, $\Delta N(\gamma) \leq R(\gamma)$, and equality is attained iff γ is a wave. This leads to the VF of LWR.

Scalar case: connection to optimal control

Mathematically, this is formulated as an optimal control problem (terminal cost problem, aka 'Bolza' problem)

$$
\begin{array}{rl}
\text{minimize} & C(x_{PQ}) = N_P + \int_{t_P}^{t_Q} r(\dot{x}_{PQ})dt \\
\text{s.t.} & \frac{d}{dt}C = q(t, x_{PQ}(t)) - \dot{x}_{PQ}(t)\rho(t, x(t)) \\
& q = f(\rho) \\
& P \in \mathcal{B};\ Q \text{ is given}
\end{array}
$$

Scalar case: Connection to HJ

Calculus of variation technique is used to derive the HJ equation from the optimal control formulation (see e.g. Naidu, 2002).

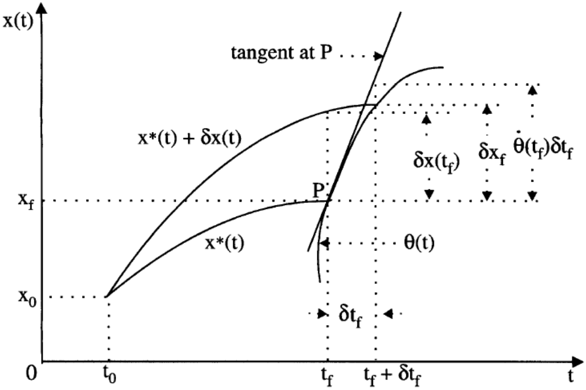
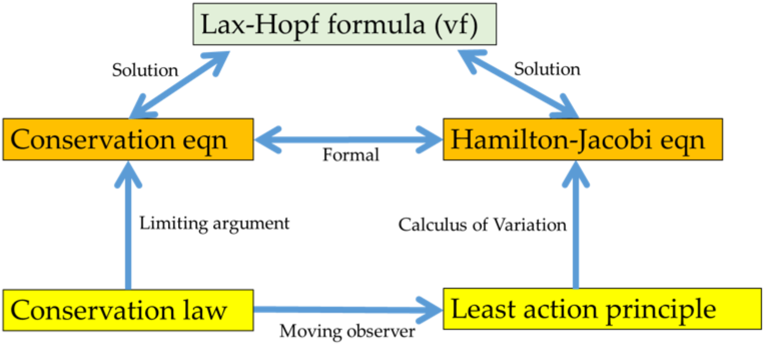


Figure 2.8 Final-Point Condition with a Moving Boundary $\theta(t)$

Idea for extension



Payne-Whitham Model (1971)

$$\partial_t v + v\partial_x v = -\frac{v - v_e(\rho)}{T} + \frac{v_e'(\rho)}{2T\rho}\frac{\partial \rho}{\partial x} \tag{3}$$

It describes a relation between acceleration and stimuli. Early second-order models usually adopt a similar form, i.e.

total derivative of speed=function of stimuli

- Complications arise when shock develops
- Receive critique from Daganzo (1995), mainly for the violation of anisotropic principle
- The issue is rectified (ARZ and others), when hyperbolic conservation law methodology is adopted in model development

Why second-order?

By construction, second-order models should capture collective effects resulted from driver's relaxation and response to gradient of traffic states ('second-order' information). Different from the LWR model, instability is a typical character of such systems.

Empirical evidence on 'irregularities' warrants modeling effort beyond the original LWR model (Helbing, 2004)

- Oscillation can magnify
- Cluster can form
- Phase transition near and after breakdown appears chaotic

Second-order modeling provides a possible way of explaining and simulating these phenomena – a sometimes controversial one. Yet, increasing data availability makes it possible to test and revise such theories.

Formulated as 2×2 hyperbolic conservation systems

Analysis of second-order models becomes a routine when its LHS is formulated as a hyperbolic conservation system (Leveque, 1992; Dafermos, 2005)

$$\begin{cases} \partial_t u_1 + \partial_x f(u_1, u_2) = 0 \\ \partial_t u_2 + \partial_x g(u_1, u_2) = 0 \end{cases}$$

Then problem amounts to finding eigenvalues and eigenvectors of $\partial(f, g)/\partial(u_1, u_2)$ and applying the standard procedure of solving Riemann problems, i.e. connecting neighboring states with proper waves (self-similar, entropy-satisfying).

It turns out major second-order models can be formulated in this way, for example–

Examples

Models adopting conservation system form

- Payne-Whitham (1974)

$$\partial_t q + \partial_x(q^2/\rho + c_0^2\rho) = \frac{\rho v_e(\rho) - q}{T}$$

- Aw-Rascle (2000)

$$\partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = 0$$

- Lebacque (2007)

$$\partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0$$

Seems an exception

- Zhang (2002)

$$\partial_t v + v\partial_x v = -c(\rho)v_x$$

A priori estimate on g

When aforementioned 2×2 system is proper to model traffic flow? We suppose $u_1 \equiv \rho$, $f \equiv \rho v = u_1 v = u_1 v(u_1, u_2)$,. There are at least two *a priori* requirements:

- Anisotropic: necessarily, $(\partial_2 v, -\partial_1 v) \cdot (\partial_1 g, \partial_2 g)' \geq v \partial_1 v$
- GSOM: if $s = \rho I(\rho, s)$, and I is constant along vehicle trajectories, then flux g satisfies $(\partial_x \rho, \partial_x s) \cdot (\partial_1 g, \partial_2 g)' = s \partial_x v + v \partial_x s$

These two combine to give

$$\|\nabla g\| \geq \max\{\frac{v v_\rho}{\|\nabla v\|}, \frac{s v_x + v s_x}{\|(\rho_x, s_x)\|}\}$$

Example of flux pair

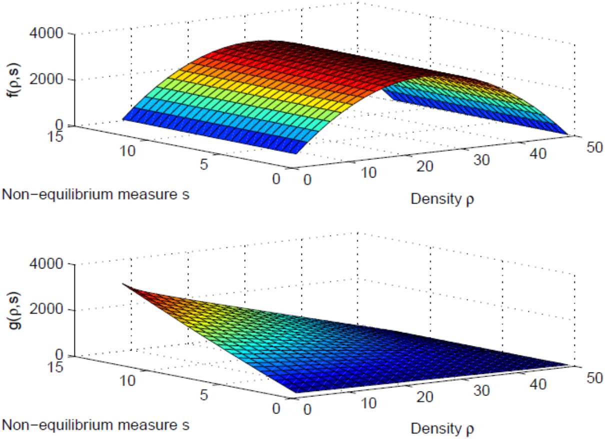
For illustration, we consider the following flux pair:

$$\begin{cases} f(\rho, s) = \rho v_e(\rho) + s \\ g(\rho, s) = v_e(\rho) s + \frac{s^2}{\rho} \end{cases}$$

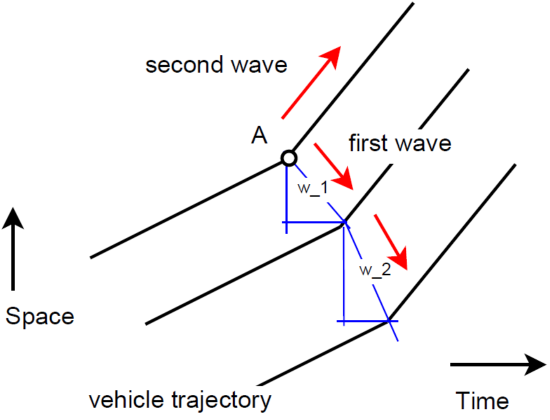
Properties

- Here s represents the deviation of actual flux from nominal flux $Q_e \equiv \rho v_e(\rho)$
- This is a special case of Lebacque (2007), with variable translation
- f and g are non-convex-concave, actually $\partial_\rho^2 f < 0$, $\partial_s^2 f = 0$, $\partial_\rho^2 g \geq 0$, $\partial_s^2 g > 0$

The flux pair



Wave


second wave
A
first wave
w_1
w_2
Space
vehicle trajectory
Time

Main theorem

When relaxation is not considered, N_ρ and N_s adopt the following variational representations

$$\begin{cases} N_\rho(t,x) = \inf\{N_\rho(t_{\eta_\rho}, \eta_\rho(t_{\eta_\rho})) + \int_{t_{\eta_\rho}}^{t} R_\rho(s(\tau,\eta_\rho(\tau)), \dot{\eta}_\rho) d\tau : \eta_\rho \text{ is a path from } \\ N_s(t,x) = \sup\{N_s(t_{\eta_s}, \eta_s(t_{\eta_s})) + \int_{t_{\eta_s}}^{t} R_s(\rho(\tau,\eta_s(\tau)), \dot{\eta}_s) d\tau : \eta_s \text{ is a path from } \end{cases} \tag{4}$$

if the following ordinary differential equations (ODEs) admit Lipschitz continuous solutions

$$\begin{cases} \dot{\eta}_\rho(\tau) = \partial_\rho f(\rho(\tau,\eta_\rho(\tau)), s(\tau,\eta_\rho(\tau))) \\ \eta_\rho(t) = x \end{cases} \tag{5}$$

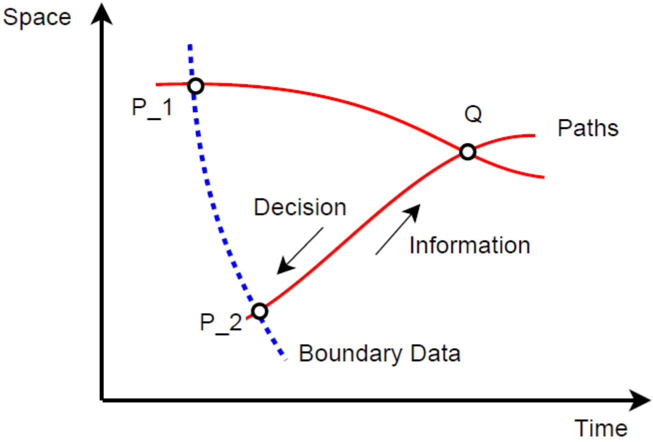
$$\begin{cases} \dot{\eta}_s(\tau) = \partial_s g(\rho(\tau,\eta_s(\tau)), s(\tau,\eta_s(\tau))) \\ \eta_s(t) = x \end{cases} \tag{6}$$

We call the solutions to these ODEs optimal paths pertaining to scalar field ρ and s respectively.

Remarks

- Sketch of Proof: we can prove by demonstrating that LHS$\geq$RHS and LHS$\leq$RHS. One direction is trivial, the other direction use the existence of solution of the given ODEs, interpretation of which is similar to scalar case
- The solution involves coupledness of the two state variables as expected, a price to pay with systems compared to scalar case
- The given ODEs involve discontinuous RHS with solution of a conservation system embeded; this type of ODEs may or may not admits a solution; in the case of LWR, corresponding ODE always adopts a solution, explaining why VF of LWR exists. See Bressan and Shen (2000) for flavors of such problem
- This problem is related to the so-called 'externality problem' investigated by Loreti and Vergara Caffarelli (2000, 2004). Solution of the ODEs can be interpreted as simultaneous coupled decisions of two persons whose dynamics are dictated by system state (ρ, s) and others decision

Wave and optimal path



Considering relaxation

We estimate the deviation of a vehicle trajectory, with and without relaxation, during $[0, t]$. The Result follow

$$|x_F(t) - \tilde{x}_F(t)| \leq \frac{l_0}{\bar{c}} \int_0^t \exp(-s/T)ds = \frac{l_0}{\bar{c}} \frac{1 - \exp(-t/T)}{T} \leq \frac{l_0}{\bar{c}T}$$

where $\bar{c}$ is the upper bound for density during the evolution. This illustrate the worst case scenario.

1. Set current time $i_0 = 0$;
2. For all (i, j) such that $i = i_0$, we calculate the approximate value of ρ and s,
$$\rho(i,j) = \sum_{|j'-j|=1} |N_\rho(i,j') - N_\rho(i,j)|/|\{j'\}|\Delta x$$
$$s(i,j) = \sum_{|j'-j|=1} |N_s(i,j') - N_s(i,j)|/|\{j'\}|\Delta x$$
3. Update N-values of ρ and s according to the variational formulas derived above
$$N_\rho(i+1,j) = \min_{j'\in A}\{N_\rho(i,j') + \Delta t R_\rho(s(i,j'),(j'-j)\Delta x/\Delta t)\}$$
$$N_s(i+1,j) = \max_{j'\in A}\{N_s(i,j') + \Delta t R_s(\rho(i,j'),(j'-j)\Delta x/\Delta t)\}$$
where $A = \{j : (i,j) \in \mathcal{B}\}$;
4. If N-values of all nodes in $\mathcal{O}$ are obtained, stop; otherwise, set $i_0 = i + 1$, $\mathcal{B} = \mathcal{B} \cup \{(i,j) : i = i_0\}$, and go to step 2.

An immediate observation

The proposed numerical scheme is exact only if $\partial_\rho f$ and $\partial_s g$ are piecewise constant, and they take values in the set $\{m\Delta x/\Delta t\}$. More generally, exact solution on grid is possible only if values of waves comprise a finite set.

Summary

This study aims to identify whether variational principle is valid for traffic flow models other than the LWR model, the form of solution, and possible applications. We derive a necessary condition for existence and analytical form of variational solution of non-equilibrium models admitting 2×2 conservation form, and discuss the effect of relaxation.

Directions to go:

- Model validation
- Modeling hybrid flow with non-equilibrium effects
- Numerical tests and error analysis