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The evening commute with cars and transit: Duality results and user equilibrium for the combined morning and evening peaks

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Abstract

This paper extends Vickrey's (1969) commute problem for commuters wishing to pass a bottleneck for both cars and transit that share finite road capacity. In addition to this more general framework considering two modes, the paper focuses on the evening rush, when commuters travel from work to home. Commuters choose which mode to use and when to travel in order to minimize the generalized cost of their own trips, including queueing delay and penalties for deviation from a preferred schedule of arrival and departure to and from work. The user equilibrium for the isolated morning and evening commutes are shown to be asymmetric because the schedule penalty in the morning is the difference between the departure and wished curves, and the schedule penalty in the evening is the difference between the arrival and wished curves. It is shown that the system optimum in the morning and evening peaks are symmetric because queueing delay is eliminated and the optimal arrival curves are the same as the departure curves.

The paper then considers both the morning and evening peaks together for a single mode bottleneck (all cars) with identical travelers that share the same wished times. For a schedule penalty function of the morning departure and evening arrival times that is positive definite and has certain properties, a user equilibrium is shown to exist in which commuters travel in the same order in both peaks. The result is used to illustrate the user equilibrium for two cases: (i) commuters have decoupled schedule preferences in the morning and evening, and (ii) commuters must work a fixed shift length but have flexibility when to start. Finally, a special case is considered with cars and transit: commuters have the same wished order in the morning and evening peaks. Commuters must use the same mode in both directions, and the complete user equilibrium solution reveals the number of commuters using cars and transit and the period in the middle of each rush when transit is used.

Keywords: morning commute, evening commute, bottleneck, cars and transit, user equilibrium

1. Introduction

The morning commute at a bottleneck with finite capacity to serve cars has been extensively studied following Vickrey (1969). This bottleneck model considers a population of commuters that wish to depart a bottleneck in order to reach a destination on time. All commuters are assumed to choose when to travel in order to minimize the sum of the costs of their own free-flow trip cost, queueing time, and penalty for schedule deviation. The unique equilibrium that results allows no commuter to reduce his or her own travel cost by unilaterally choosing another arrival and departure position at the bottleneck (Smith, 1984; Daganzo, 1985).

The reverse problem in the evening has been addressed explicitly in only a few studies (Fargier, 1981; DePalma and Lindsey, 2002), which recognize that the evening commute differs from the morning but are limited to an isolated evening rush involving only cars. Vickrey (1973) and DePalma and Lindsey (2002) state that evening commute is the mirror image of the morning commute unless the commuters are heterogeneous. However, this paper shows that this is not the case, even with identical or mirrored values of earliness and lateness in the morning and evening. If evening commuters also seek to minimize the cost of their own trip, then the user equilibrium for the evening must be a pattern of bottleneck arrivals and departures that allow no commuter to reduce his or her own cost by choosing another arrival position at the bottleneck.

This paper addresses both the morning and evening commutes with a more general framework for cars and a collective mode such as transit. In addition to the choice of when to travel, commuters are able to choose which mode to use. Although mode choice has been studied in a number of works for the morning commute (Tabuchi, 1993; Braid, 1996; Huang, 2000; Danielis and Marcucci, 2002; Qian and Zhang, 2011), none considers the evening commute. These works also assume that the transit service is parallel to the road where the bottleneck resides. The bi-modal analysis in this paper follows closely from Gonzales and Daganzo (2012) which presents a user equilibrium solution for the morning commute where transit service shares the bottleneck capacity with cars, for example by dedicating a lane to high occupancy vehicles. An additional improvement in this paper is that the transit mode is considered to have finite capacity.

Although studying the morning and evening commutes in isolation provides some interesting insights, the reality is that commuters make travel decisions based on their schedule for the whole day. Existing works that seek to model daily travel decisions at a bottleneck are limited, and what studies exist rely on linking the morning and evening commutes via work duration (Zhang et al., 2005) or parking availability (Zhang et al., 2008) and consider only the use of cars. There is a need for understanding how commuters make daily travel decisions with more general schedule preferences and considering the role that mode choice plays in the dynamics of daily travel choices.

The paper is organized as follows. Section 2 describes the user equilibrium for the morning and evening peaks in isolation. First, the well known user equilibrium solution for the morning commute is reviewed and extended to consider transit service with finite capacity. Then, the user equilibrium for the evening is presented and shown to be different from the result for the morning peak. Section 3 describes the duality of the system optimum for the isolated morning and evening peaks. Finally, Section 4 presents the user equilibrium findings for the combined morning and evening peaks if only cars are used, followed by a special case in which the combined user equilibrium can be easily identified when commuters are able to choose between cars and transit.

2. Asymmetry of Morning and Evening User Equilibrium

The user equilibrium problem for the morning peak has been extensively studied following Vickrey (1969). The morning commute at a bottleneck serving cars and uncapacitated transit is presented in detail in Gonzales and Daganzo (2012). In Section 2.1 of this paper, we present the morning user equilibrium solution considering that transit may have a finite capacity. When no transit is operated, the bottleneck carries only cars, and the person-carrying capacity is μ. When transit is operated, the person-carrying capacity of the bottleneck is the combined capacity for cars and transit vehicles, μ_o. If the transit service is provided with a dedicated right of way, then μ_o reflects the sum of the capacity of the lanes for cars and the passenger-carrying capacity of the transit service. Then, in Section 2.2 we present the user equilibrium for the evening commute that occurs when commuters experience a schedule penalty based on when they choose to leave work and arrive at the same bottleneck in the evening. Finally, in Section 2.3 we discuss the benefit of providing transit in the morning and evening peaks, as well as the effect of transit's capacity constraint.

2.1. Morning Peak with Finite Transit Capacity

Following the formulation of the morning bottleneck problem in Vickrey (1969) and the extension to two modes in Gonzales and Daganzo (2012), we consider a population of N commuters that wish to depart a bottleneck in order to arrive at a destination on time. These wished times can be described by a cumulative count of commuters that wish to depart the bottleneck by time t in the morning, $W_m(t)$. These commuters are identical in their preferences except for their wished departure time. Each commuter chooses when to arrive at the bottleneck and which mode to use (car or transit) in order to minimize the generalized cost of his or her own trip including queueing time and schedule deviation. The resulting cumulative arrival curve, $A_m(t)$, and departure curve, $D_m(t)$, are the user equilibrium travel pattern for the morning commute.

In order to provide simple closed form solutions, we will consider the special case that $W_m(t)$ is Z-shaped with slope $\lambda_m \geq \mu_o$. We will also consider a bilinear schedule penalty such that commuters experience each minute of early departure as $0 < e < 1$ minutes of queueing time and each minute of late departure as $L > 0$ minutes of queueing time.[1]

The equilibrium arrival curve must have a slope, $\dot{A}_m(t)$, that provides no incentive for a commuter to reduce the generalized cost of his or her commute by choosing another arrival position. This slope depends on the departure rate that each commuter experiences at the bottleneck. At equilibrium, an early commuter choosing to depart Δt later will reduce his or her schedule penalty by $e\Delta t$. If the departure rate from the bottleneck is μ, then the queueing delay will increase by $\Delta t - \mu\Delta t/\dot{A}_m(t)$, and at equilibrium this must exactly equal the reduction in schedule penalty. A similar condition must hold for late commuters. Zhang et al. (2010) extend Vickrey's equilibrium solution to consider bottlenecks with time-dependent capacity. In the case of the morning commute with transit, the capacity depends on when transit service is operated, so the slope of the equilibrium arrival curve follows a similar form:

$$\dot{A}_m(t) = \begin{cases} \mu/(1-e) & \text{if commuters are early, only cars are used} \\ \mu_o/(1-e) & \text{if commuters are early, cars and transit are used} \\ \mu_o/(1+L) & \text{if commuters are late, cars and transit are used} \\ \mu/(1+L) & \text{if commuters are late, only cars are used} \end{cases} \tag{1}$$

as illustrated in Figure 1.

If only cars are used, then the equilibrium takes the unique form identified in Smith (1984) and Daganzo (1985), and the maximum delay is

$$T_{\max,car} = \frac{NeL}{\mu(e+L)}, \tag{2}$$

experienced by the single commuter who departs the bottleneck on time. However, if commuters are able to choose an alternative transit service with generalized cost per user for an uncongested trip $z_T > z_C$, where z_C is the generalized cost of an uncongested car trip, then transit will be competitive when the queueing delay for cars reaches $T_T = z_T - z_C$. This definition of transit broadly represents many types of collective passenger transportation modes, such as carpool lanes, which increase the passenger capacity of the bottleneck while imposing some fixed cost or penalty on users for the inconvenience of using the higher-occupancy vehicle.

At the beginning and end of the rush, only cars are used while the total cost of a car commute including queueing delay is less than that of a free-flow transit trip. Once the car delay grows to $T_T = z_T - z_C$, then commuters become indifferent between transit and car, so both modes will be used. When $\lambda_m > \mu_o$, the demand exceeds the combined capacity of the bottleneck for cars and transit. If $\lambda_m \leq \mu_o$, then all trips could be served as they arrive (i.e., all trips would be served on time), and the delay for cars would hold steady at T_T, as shown in Gonzales and Daganzo (2012).

We consider the morning commute in three parts: commuters at the beginning of the rush that only drive, commuters in the middle of the rush that use cars and transit, and commuters at the end of the rush that only drive. For convenience, we use points labeled in Figure 1 with capital letters to denote important values.

[1] In much of the bottleneck literature separate cost coefficients are considered for queueing time, α, schedule earliness, β, and schedule lateness, γ. The definitions in this paper are equivalent with $e = \beta/\alpha$ and $L = \gamma/\alpha$.

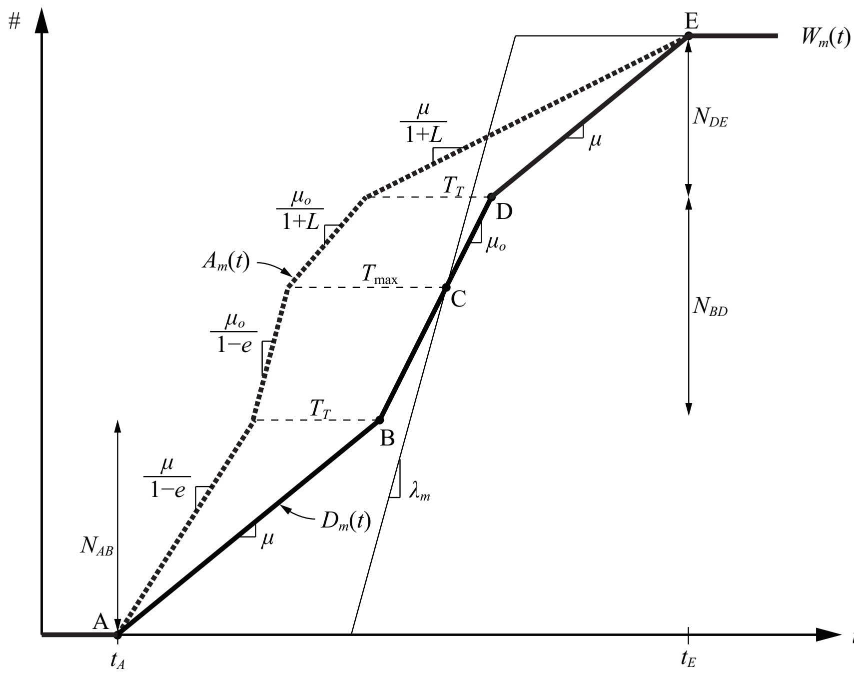

Fig. 1. User equilibrium arrival and departure curves for the morning commute with cars and transit.

For example, t_A is the time associated with point A, and $N_{AB} = N_A - N_B$ is the total number of commuters to pass between points A and B.

Let the segment $\overline{\text{AB}}$ denote the departures of early commuters at rate μ when the queueing delay is less than T_T and transit is not yet used. At point B, the queueing delay is equal to T_T, so the number of early drivers that travel before transit service begins is:

$$N_{AB} = \mu T_T / e. \tag{3}$$

Likewise, the segment $\overline{\text{DE}}$ denotes the departures of late commuters at rate μ when the queuing delay is less than T_T and transit is no longer used. The number of late drivers that travel after transit service ends is:

$$N_{DE} = \mu T_T / L. \tag{4}$$

This leaves the remaining commuters to depart the bottleneck in the middle of the rush at rate μ_o when the queueing delay exceeds T_T so transit is competitive and both modes are used simultaneously. By subtracting (3) and (4) from the total number of commuters and combining terms, the number of mid-rush travelers is:

$$N_{BD} = N - \frac{\mu T_T (e + L)}{eL}. \tag{5}$$

The equilibrium arrival curve must satisfy the required slopes from (1) and make the arrival and departure curves starting at A to meet again at E as shown in Figure 1. With the early and late drivers accounted for, the equilibrium arrival and departure curves in the middle of the rush become another bottleneck problem where N_{BD} commuters experience queueing in addition to T_T. Even though some of the additional costs experienced by transit riders are not in the form of travel time, it is expressed graphically as such so that the arrival and departure curves in Figure 1 account for all of the costs experienced by the users. Following (2), the maximum additional delay is $N_{BD} eL / \mu_o (e + L)$. So, substituting (5), the maximum total queueing delay is:

$$T_{max} = T_T + \left(N - \frac{\mu T_T (e + L)}{eL} \right) \frac{eL}{\mu_o (e + L)} \tag{6}$$

The solution determines the point C where the commuter with the maximum queuing delay departs the bottleneck on time.

The ratio of early and late commuters is the same when only cars are used and when cars and transit are used because of the required slopes of the arrival curves for all early and late commuters. Following from (3) and (4) this ratio is:

$$\frac{N_{AB}}{N_{DE}} = \frac{N_{BC}}{N_{CD}} = \frac{L}{e}, \tag{7}$$

which is the same relationship identified in Vickrey (1969).

2.2. Evening Peak with Finite Transit Capacity

Usually the evening commute is assumed to be the mirror of the morning commute. A couple of papers have addressed the evening commute explicitly for a bottleneck serving only cars (Fargier, 1981; DePalma and Lindsey, 2002). We consider an evening commute problem that is similar to Vickrey's bottleneck model for the morning commute except that commuters experience a schedule penalty associated with their choice of arrival time at the bottleneck, which depends on when they choose to leave from work. For generality, the evening bottleneck has a passenger capacity of μ' when only cars are used and μ'_o when both cars and transit use the bottleneck. These are not necessarily the same capacities as in the morning. In order to provide simple closed form solutions, we consider a case similar to that of Section 2.1 in which N commuters wish to travel past the bottleneck with a Z-shaped wished curve, $W_e(t)$ that has slope $\lambda_e \geq \mu'_o(1+e')$. Commuters experience a cost for their trip associated with queuing delay and a schedule penalty for their choice of arrival time at the bottleneck: earliness penalty, $e' > 0$, and lateness penalty, $0 < L' < 1$. We will first present the user equilibrium solution if only cars are used, and then the solution is extended to consider a bottleneck that can serve commuters at rate μ' when only cars are used and at rate μ'_o when cars and transit are used.

Since the schedule penalty is measured relative to the arrival curve instead of the departure curve, the equilibrium slopes must take a different shape. As in the morning commute, the slope of the equilibrium arrival curve, $\dot{A}_e(t)$, must allow no commuter to reduce the cost of his or her own trip by choosing another travel time. At equilibrium, an early commuter choosing to arrive Δt later will reduce his or her schedule penalty by $e\Delta t$. The queueing delay will increase by $\dot{A}_e(t)\Delta t/\mu' - \Delta t$, and at equilibrium this must exactly equal the reduction in schedule penalty. The slope of equilibrium arrival curve in the evening must satisfy:

$$\dot{A}_e(t) = \begin{cases} \mu'(1+e') & \text{if commuters are early, only cars are used} \\ \mu'_o(1+e') & \text{if commuters are early, cars and transit are used} \\ \mu'_o(1-L') & \text{if commuters are late, cars and transit are used} \\ \mu'(1-L') & \text{if commuters are late, only cars are used} \end{cases} \tag{8}$$

so that commuters have no incentive to choose another arrival time. The resulting equilibrium arrival and departure curves are illustrated in Figure 2. Note that in the evening the on time commuter corresponds to the point where $A_e(t)$ intersects $W_e(t)$.

If the bottleneck is used only by cars, then the capacity is μ' throughout the rush and points B′, C′, and D′ are all at the same position. There is a unique starting point A′ such that following the equilibrium slopes prescribed above, there will be a single point E′ where $A_e(t)$, $D_e(t)$, and $W_e(t)$ all meet again at the end of the rush. As in the morning, the longest queuing time, $T'_{\max,car}$, is experienced by the on time commuter, and the number of early and late commuters must satisfy:

$$N_{A'B'} = \frac{T'_{\max,car}\mu'(1+e')}{e'} \tag{9}$$

$$N_{D'E'} = \frac{T'_{\max,car}\mu'(1-L')}{L'}. \tag{10}$$

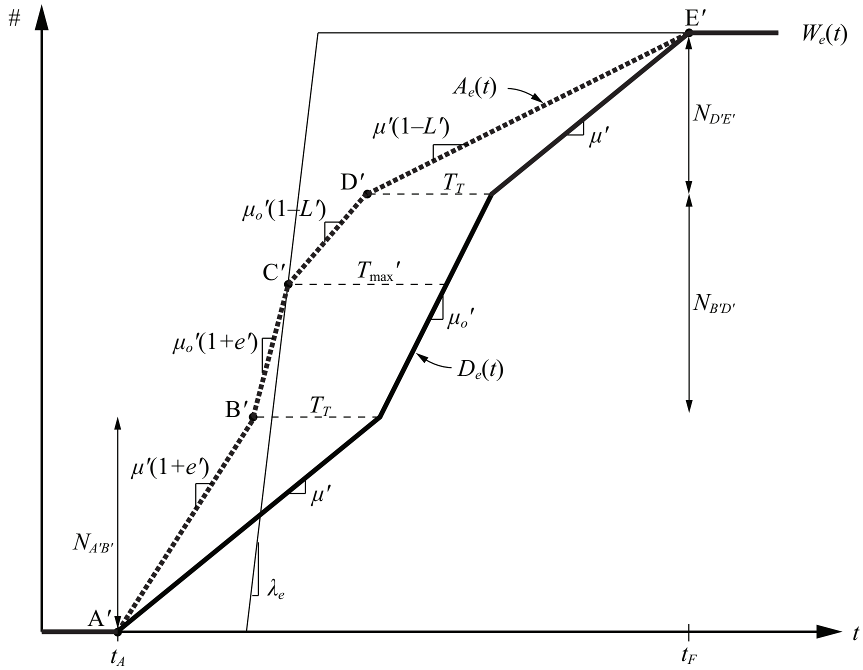

Fig. 2. User equilibrium arrival and departure curves for the evening commute with cars and transit.

Clearly, there is an asymmetry between the morning and evening commutes, because the ratio of early and late commuters does not equal that for the morning:

$$\frac{N_{A'B'}}{N_{D'E'}} = \frac{L'(1+e')}{e'(1-L')}. \tag{11}$$

The reason for the asymmetry is that earliness and lateness are measured relative to the departure curve from the bottleneck in the morning and relative to the arrival curve at the bottleneck in the evening. This difference changes the shape of the user equilibrium even if commuters are identical. This example shows that the user equilibrium for the morning and evening commutes exhibit specific intricacies that should be studied as such.

The maximum queuing delay can be obtained by substituting (9) and (10) into $N = N_{A'B'} + N_{D'E'}$, and solving for $T'_{\max,car}$. The resulting cost is similar to the morning commute:

$$T'_{\max,car} = \frac{Ne'L'}{\mu'(e'+L')}. \tag{12}$$

Although this means that the total cost of the morning and evening commutes are the same if $e = e'$ and $L = L'$, we might expect workers to commute to and from work shifts with specific start and end times. Therefore, it is more reasonable to assume that $e < e'$ and $L > L'$. Note that if the lateness penalty diminishes toward zero, the amount of queuing decreases and commuters are more likely to stay late at work or linger in the neighborhood in order to avoid the cost of queuing.

Suppose that a transit service is operated that provides the bottleneck with a combined capacity to serve commuters at rate μ_o'. The transit service will be used when the queueing delay for cars reaches $T_T = z_T - z_C$ at which point transit becomes an attractive alternative. Just as in the morning commute, we consider the evening commute in three parts: commuters are the beginning of the rush that only drive, commuters in the middle of the rush that use cars and transit, and commuters at the end of the rush that only drive.

Let the segment $\overline{\text{A'B'}}$ denote the arrivals of early commuters at rate $\mu'(1+e')$ when the queueing delay is less than T_T. Similarly, segment $\overline{\text{D'E'}}$ denotes the arrivals of late commuters at rate $\mu'(1-L')$ when the queueing delay is also less than T_T. The number of early and late commuters when cars are not used is obtained by evaluating (9) for $N_{A'B'}$ and (10) for $N_{D'E'}$ with T_T in place of $T'_{\max,car}$. Then, by subtracting these values from the total number of travelers, the number of commuters traveling in the middle of the rush when both cars and transit are used is:

$$N_{B'D'} = N - \frac{\mu' T_T(e' + L')}{e'L'}. \tag{13}$$

Note that the expression for $N_{B'D'}$ takes the same form as (5) for N_{BD}.

Just as for the morning commute, the remaining equilibrium arrival curve in the middle of the rush becomes another bottleneck problem where $N_{B'D'}$ commuters experience queueing in addition to T_T. The arrival and departure curves in Figure 2 account for the cost experienced by transit riders as well as car commuters. Following (12), the maximum additional delay is $N_{B'D'}e'L'/\mu'_o(e' + L')$. So, substituting (13), the maximum total queueing delay in the evening is:

$$T'_{\max} = T_T + \left(N - \frac{\mu' T_T(e' + L')}{e'L'}\right)\frac{e'L'}{\mu'_o(e' + L')}, \tag{14}$$

which is also the same form as (6) for $T_{\max}$. This solution determines the position of C′ as shown in Figure 2.

2.3. Benefit of Transit Service

In the preceding sections, the user equilibrium solution for the morning and evening commutes have been presented when a bottleneck serves both cars and transit. We now consider the total social benefit of providing transit service during the peak and how this benefit is affected by the capacity of the transit service. The benefit of transit is the difference between the total cost of the commute with no transit and the cost of the same commute when a transit service is available for people to choose. As in the previous section, we consider wished curves in the morning and evening peaks that have slopes $\lambda_m \geq \mu_o$ and $\lambda_e \geq \mu'_o(1 + e')$.

Considering the morning commute, (7) gives us that the number of early and late travelers is the same whether or not transit service is provided. Therefore, the total social cost for λ_m compared to the case if all commuters share an identical wished departure time is a reduction in schedule penalty between $W(t)$ and a vertical line through the critical departure time at C. From the geometry of the Z-shaped wished curve, the reduced earliness cost is $eN^2_{AC}/2\lambda_m$, and the reduced lateness cost is $LN^2_{CE}/2\lambda_m$. Since this difference is unaffected by the provision of transit service as long as $\lambda_m \geq \mu_o$, the benefit of providing a bottleneck capacity μ_o with transit service may be evaluated for the simpler case that all commuters wish to travel at the same time.

If all commuters have the same wished departure time preference, then every commuter will experience an identical cost in equilibrium. The total social cost of the morning commute, not counting the cost of a free flow car trip, is given by $NT_{\max,car}$ when no transit is operated and $NT_{\max}$ when transit is operated. Substituting from (2) and (6), the benefit of providing transit service to make the bottleneck capacity μ_o is:

$$B(\mu_o) = \frac{N^2 eL}{\mu(e + L)} - N\left(T_T + \left(N - \frac{\mu T_T(e + L)}{eL}\right)\frac{eL}{\mu_o(e + L)}\right). \tag{15}$$

If transit had no capacity constraint, and all passengers could be served at the rate they arrive, commuters in the middle of the rush would depart the bottleneck at rate λ_m. The benefit of transit with unconstrained capacity would be the same as if $\mu_o = \lambda_m$, that is to say (15) is evaluated for $B(\lambda_m)$.

It turns out that the benefit of constrained transit is always related to the benefit of unconstrained transit by the following expression:

$$B(\mu_o) = \frac{1 - \mu/\mu_o}{1 - \mu/\lambda_m} B(\lambda_m), \tag{16}$$

which can be verified by comparing $B(\mu_o)$ and $B(\lambda_m)$ using (15).

In the evening, transit has a very similar effect as long as $\lambda_e \geq \mu'_o(1 + e')$. Using $T'_{\max,car}$ from (12) and $T'_{\max}$ from (14) instead of the values from the morning commute, the expression for $B(\mu_o)$ takes the same form except that e' replaces e, L' replaces L, and λ_e replaces λ_m. As a result, (16) also applies to the evening commute. This result is useful, because it means that we can easily calculate the effect of changing transit capacity without having to resolve the entire user equilibrium.

3. Symmetry of Morning and Evening System Optimum

Up to this point we have considered the user equilibrium for the morning and evening peaks and highlighted some of the asymmetries between the two peaks. We now turn our attention to the system optimum. The system optimal arrival and departure curves are those which minimize the total social cost of the commute assuming that people choose when to travel and which mode to use for the common good.

Although the user equilibrium involves queueing any time the demand exceeds the bottleneck capacity, there should be no queueing in the system optimum because the queue can always be eliminated by forcing the arrival curve to take the same value as the departure curve. A well known property of the morning commute with a single mode is that the optimal fine toll is the one which charges each departing vehicle an equal value to the cost they would experience as queueing delay in equilibrium. As a result, the commuters will exchange their queueing cost for toll cost and choose to arrive at the time when they are able to be served (Vickrey, 1969). The system optimum solution for the morning peak with cars and transit was extended in Gonzales and Daganzo (2012), and the result is that the system optimum can take one of the three forms: all trips served by car, all trips served by transit, or trips are served by a mix of cars and transit where transit is used in the middle of the rush period.

Whereas the user equilibrium solutions for the morning and evening are not symmetric, as discussed in the preceding section, there is a duality relation between the system optimum in the morning and evening peaks. In the morning $W_m(t)$ is the wished departures from the bottleneck, and in the evening $W_e(t)$ is the wished arrivals at the bottleneck. In the system optimum, when queueing has been eliminated, the arrival curve is the same as the departure curve, so the relationship between W and D is the same in the morning and evening. Therefore, the solution to the morning commute system optimum problem is also the solution to the evening problem.

Since the system optimum fine toll for the morning commute with a single mode and with both cars and transit have already been developed (Arnott et al., 1990; Gonzales and Daganzo, 2012), we will not repeat the full solutions here. However, it is worth pointing out that there are some implications for pricing. The optimal toll in the morning should increase at at rate e while commuters are early and decrease at rate $-L$ while commuters are late. This is the same as charging commuters the equivalent of the cost time they would have spent queuing in equilibrium. In the evening, the optimal toll should increase at rate e' while commuters are early and decrease at rate $-L'$ while commuters are late. The evening toll is not the same as the queueing delay that would have been experienced in equilibrium. The difference is due to the fact that the schedule deviation in the evening is measured with respect to the arrival curve, and the arrival curve keeps changing as the tolls change.

Another useful insight is that the optimal prices in the morning and evening peaks can be used to eliminate queueing in congested street networks. Gonzales and Daganzo (2012) demonstrates that by eliminating queueing in a network, optimal prices allow us to maintain the vehicle flow in a street network at capacity as if the street network had a fixed capacity like a bottleneck. By preventing gridlock conditions, optimal pricing of street networks is even more beneficial than optimal pricing at isolated bottlenecks. Since the optimal prices in the evening commute have the same effect on eliminating queuing congestion, the optimal evening tolls can also be implemented on congestable street networks.

4. User Equilibrium for the Combined Morning and Evening Commutes

Although isolated morning and evening rushes are interesting, our goal is to understand both peaks together since this is how we believe people make daily travel decisions. What little work exists on this subject (Zhang et al., 2005, 2008) links the morning and evening commutes via work duration and parking availability. These assumptions are relaxed in this paper to include mode choice and more general linkages for cases with identical commutes; i.e., with the same wishes and schedule penalty functions. The linkages are expressed by means of schedule penalties $S(d_m, a_e)$ that are functions of two variables: a commuter's departure time from the bottleneck in the morning, d_m, and their arrival time to the bottleneck in the evening, a_e. We consider S functions that are positive definite, twice differentiable function with partial derivatives such that $\partial S/\partial d_m > -1$, $\partial S/\partial a_e < 1$, and $\partial^2 S/\partial d_m \partial a_e \leq 0$. These conditions imply that earliness in the morning and lateness in the evening are less costly than queueing delay. A schedule penalty function may be defined based on the difference between the time-dependent utility of being at work and the time-dependent utility of being at home as described in Ettema and Timmermans (2003) and Zhang et al. (2005). In this paper, the schedule penalty is the opportunity cost expressed in units of equivalent queueing delay for a commuter not being at his or her preferred location.

Proposition 1 (User Equilibrium for Cars in Combined Morning and Evening Commutes)**.** *If $S(d_m, a_e)$ is a positive definite, twice differentiable function with partial derivatives such that $\partial S/\partial d_m > -1$, $\partial S/\partial a_e < 1$, and $\partial^2 S/\partial d_m \partial a_e \leq 0$, then a user equilibrium for a car-only bottleneck exists for the combined morning and evening peaks in which the commuters depart and arrive in the same first-in-first-out (FIFO) order in both peaks.*

Proof. We define the arrival times of commuters at the bottleneck in the morning and evening as $a_m(m)$ and $a_e(n)$, where m and n are the arrival positions in morning and evening. For a bottleneck with FIFO queue discipline, the departure order is the same as the arrival order, and the departure times at the bottleneck are $d_m(m)$ and $d_e(n)$.

The departure curves in the morning and evening are determined by the arrival curves. The first vehicle in each peak is served without delay because no queue has formed, so $d_m(0) = a_m(0)$ and $d_e(0) = a_e(0)$. Since the bottleneck serves trips at capacity μ during the morning peak and μ' during the evening peak, the departure curves are given by:

$$d_m(m) = a_m(0) + m/\mu \tag{17}$$
$$d_e(n) = a_e(0) + n/\mu'. \tag{18}$$

Thus, an equilibrium with the same order in the morning and evening is completely defined by the arrival curves. The generalized cost for a commuter in positions m and n is the sum of the schedule penalty and the queuing time in each peak:

$$Z(m,n) \doteq S\,(d_m(m), a_e(n)) + d_m(m) - a_m(m) + d_e(n) - a_e(n). \tag{19}$$

In order for the arrival curves to be an equilibrium of this type, $Z(m,n)$ must reach a global minimum with respect to m when $m = n$ and with respect to n when $n = m$; i.e., $Z(m,n) \geq Z(n,n), Z(m,m)$. This is done in two steps: (i) setting up the first order conditions and showing that there is a unique set of arrival curves that satisfy them; and (ii) verifying that the solution is a global minimum.

The necessary first order conditions for the cost to reach a global minimum for a specific value of m and n are obtained by substituting (17) and (18) into (19):

$$\frac{\partial Z}{\partial m} = \frac{\partial S}{\partial d_m}\frac{dd_m(m)}{dm} + \frac{1}{\mu} - \frac{da_m(m)}{dm} = 0 \tag{20}$$
$$\frac{\partial Z}{\partial n} = \frac{\partial S}{\partial a_e}\frac{da_e(n)}{dn} + \frac{1}{\mu'} - \frac{da_e(n)}{dn} = 0. \tag{21}$$

The equilibrium arrival times a_m and a_e must satisfy (20) and (21) for all $m = n$ so that the same departure order in the morning and evening yields the minimum generalized cost to each commuter. Following from (20) and (21) are a pair of coupled ordinary differential equations:

$$\frac{da_m(n)}{dn} = \frac{1}{\mu}\left(\frac{\partial S}{\partial d_m} + 1\right) \tag{22}$$
$$\frac{da_e(n)}{dn} = -\frac{1}{\mu'}\left(\frac{\partial S}{\partial a_e} - 1\right)^{-1} \tag{23}$$

with mixed initial/final conditions:

$$a_m(N) - a_m(0) = N/\mu \tag{24}$$
$$a_e(N) - a_e(0) = N/\mu'. \tag{25}$$

The unique solution of (22), (23), (24), and (25) identifies the arrival curves. Note that they are increasing.

It remains to be shown that $Z(m,n) \geq Z(n,n)$ and $Z(m,m)$ for all $m \neq n$. The proof of both inequalities is very similar and therefore only the first is given. Consider the difference in generalized cost for a commuter who arrives at position n in the evening rush but at position m in the morning rush compared to a commuter who travels at position n in both rushes. By substituting (17) and (18) into (19), evaluating $Z(m,n) - Z(n,n)$, and simplifying, this difference may be expressed as:

$$Z(m,n) - Z(n,n) = S\left(d_m(m), a_e(n)\right) - S\left(d_m(n), a_e(n)\right) + \frac{1}{\mu}(m-n) - \left(a_m(m) - a_m(n)\right). \tag{26}$$

We will make use of two expressions to show that $Z(m,n) - Z(n,n) \geq 0$. For simplicity of notation, we use subscripts m and e (in this proof only) to denote the partial derivatives of S with respect to its first and second arguments. Now consider the last term of (26), the difference in morning arrival time, which can be calculated using the differential equation from (22) evaluated at the local minimum:

$$a_m(m) - a_m(n) = \int_n^m \frac{da_m(x)}{dn}dx = \frac{1}{\mu}\int_n^m S_m\left(a_m(0) + \frac{x}{\mu}, a_e(x)\right)dx + \frac{1}{\mu}(m-n). \tag{27}$$

Now consider the first term of (26), which is:

$$S\left(d_m(m), a_e(n)\right) - S\left(d_m(n), a_e(n)\right) = \frac{1}{\mu}\int_n^m S_m\left(a_m(0) + \frac{x}{\mu}, a_e(n)\right)dx. \tag{28}$$

By substituting (27) and (28) into (26) and canceling terms, we see that:

$$Z(m,n) - Z(n,n) = \frac{1}{\mu}\int_n^m \left\{S_m\left(a_m(0) + \frac{x}{\mu}, a_e(n)\right) - S_m\left(a_m(0) + \frac{x}{\mu}, a_e(x)\right)\right\}dx \geq 0. \tag{29}$$

The last inequality holds because $S_{me} \leq 0$, so that S_m is non-increasing in its second argument. Thus, $Z(m,n) \geq Z(n,n)$ for all n.

Confirmation that $Z(m,n) \geq Z(m,m)$ follows a similar procedure. We begin by using (19) to evaluate $Z(m,n) - Z(m,m)$:

$$Z(m,n) - Z(m,m) = S\left(d_m(m), a_e(n)\right) - S\left(d_m(m), a_e(m)\right) + \frac{1}{\mu'}(n-m) - \left(a_e(n) - a_e(m)\right). \tag{30}$$

The last term of (30) is the difference in evening arrival time. It can be evaluated using the differential equation from (23), which can be equivalently expressed as $da_e(n)/dn = 1/\mu'(1 - S_e)$, where S_e is evaluated at the local minimum:

$$a_e(n) - a_e(m) = \int_m^n \frac{da_e(x)}{dn}dx = \frac{1}{\mu'}\int_m^n \frac{1}{1 - S_e\left(a_m(0) + \frac{x}{\mu}, a_e(x)\right)}dx. \tag{31}$$

By expressing the first terms of (30) as integrals from m to n and combining terms, these simplify to the following expression:

$$\begin{aligned} S\left(d_m(m), a_e(n)\right) - S\left(d_m(m), a_e(m)\right) + \frac{1}{\mu'}(n-m) &= \frac{1}{\mu'}\int_m^n \frac{S_e\left(a_m(0) + \frac{m}{\mu}, a_e(x)\right)}{1 - S_e\left(a_m(0) + \frac{m}{\mu}, a_e(x)\right)}dx + \frac{1}{\mu'}\int_m^n 1dx \\ &= \frac{1}{\mu'}\int_m^n \frac{1}{1 - S_e\left(a_m(0) + \frac{m}{\mu}, a_e(x)\right)}dx. \end{aligned} \tag{32}$$

Note that the denominator is the result of substitution of $da_e/dn = 1/\mu'(1 - S_e)$. The evaluation of (30) is equivalent to subtracting (31) from (32):

$$Z(m,n) - Z(m,m) = \frac{1}{\mu'}\int_m^n \left\{\frac{1}{1 - S_e\left(a_m(0) + \frac{m}{\mu}, a_e(x)\right)} - \frac{1}{1 - S_e\left(a_m(0) + \frac{x}{\mu}, a_e(x)\right)}\right\}dx \geq 0. \tag{33}$$

The inequality holds because $S_{em} \leq 0$, so S_e is non-increasing in the first argument, and therefore $1/(1-S_e)$ is also non-increasing. Thus $Z(m,n) \geq Z(m,m)$ is shown for all m as well. □

Proposition 1 proves the existence of a user equilibrium for a formulation based on a fairly general schedule penalty function, $S(d_m, a_e)$. This includes separable penalty functions of the forms: (i) $S(d_m, a_e) = S_m(d_m) + s[a_e - d_m] + S_e(a_e)$, and (ii) $S(d_m, a_e) = S_c(a_e - d_m)$, where the S functions of type described in Proposition 1. Case (i) includes situations where morning and evening decisions are decoupled. Case (ii) captures situations where users have the flexibility to arrive early or late, subject to a fixed work-shift duration $a_e - d_m = \Delta$. The proposition also applies to linear combinations of (i) and (ii) and to piecewise differentiable schedule penalty functions; e.g., when the S functions are bilinear, since as the reader can verify, the proof holds verbatim for this case too. Section 4.1 presents the closed form user equilibrium solutions for two typical cases when the S functions are bilinear penalty.

4.1. Independent Morning and Evening Schedule Preferences

Consider a combined morning and evening commute in which all N identical commuters must pass a bottleneck by car. The schedule penalty for each commuter is $S(d_m, a_e) = S_m(d_m) + S_e(a_e)$, where $S_m(d_m)$ and $S_e(a_e)$ are bilinear with earliness penalty e and e' and lateness penalty L and L' as described in Sections 2.1 and 2.2. Proposition 1 gives us that for this schedule penalty function there is a user equilibrium with commuters traveling in the same order in both peaks. Therefore, the user equilibrium for the morning and evening peaks can be solved independently such that each commuter chooses the same position in the morning and evening.

The equilibrium arrival curves are based on the slopes of the schedule penalty function for early and late commuters. In the morning and evening, by evaluating (22) and (23) with S as defined above, the slope of the arrival curves with respect to queuing position must be:

$$\frac{da_m(m)}{dm} = \begin{cases} \frac{1}{\mu}(1-e) & \text{for early commuters} \\ \frac{1}{\mu}(1+L) & \text{for late commuters} \end{cases} \tag{34}$$

$$\frac{da_e(n)}{dn} = \begin{cases} -\frac{1}{\mu'}(-e'-1)^{-1} & \text{for early commuters} \\ -\frac{1}{\mu'}(L'-1)^{-1} & \text{for late commuters} \end{cases}. \tag{35}$$

By inverting each of these expressions, we get the same slopes for the cumulative arrival curves with respect to time as derived for the isolated morning and evening peaks (1) and (8).

The optimal prices that eliminate queueing for this combined commute are the same as the solution for the separate morning and evening commutes. These achieve the system optimum as presented in Section 3. In general, the time dependent prices in the morning are not the same as those in the evening because they depend on the values of earliness and lateness. Only in the case that $e = e'$ and $L = L'$ are the optimal tolls exactly the same in both peaks.

4.2. Rigid Work Duration

Consider the same N identical commuters using a bottleneck in the morning and evening with cars only. Now suppose that their schedule preference is for a work shift of rigid duration Δ, but commuters have some flexibility to choose when to work their shift, so they have a preferred start and end time that are Δ hours apart. Then the schedule penalty takes the form $S(d_m, a_e) = S_m(d_m) + S_c(a_e - d_m) + S_e(a_e)$ where $S_c = \infty$ if $a_e - d_m < \Delta$, and $S_c = 0$, otherwise. This function prohibits commuters from working a shorter shift. The S_m and S_e functions are bilinear as defined above. The solution can be developed for general cases with any value of $e + e'$, μ, and μ', e.g., using (20) and (21). When $e + e' < 1$ and $\mu \leq \mu'$, a very simple solution exists, which is presented here. These conditions are not totally unreasonable. They include the case that a bottleneck has symmetric capacity in the morning and evening, and the flexibility of work-shift scheduling may allow commuters to experience a low earliness penalty.

In the morning rush, commuters cannot pass the bottleneck at a rate greater than the capacity, μ. The departure curve from the bottleneck in the morning determines the time when each commuter arrives at work to start his or her shift. The work shifts end at a time Δ later, and the commuters are able to leave work in the evening and arrive at the bottleneck at the same rate they departed, so $\dot{A}_e(t) = \mu$. The rigid shift links the morning and evening commutes so that there should be no congestion at the bottleneck in the evening, and therefore $A_e(t) = D_e(t)$. Any penalty associated with passing the bottleneck early or late in the evening is reflected in the travel decision in the morning. Therefore, the problem simplifies to a morning commute formulation with a schedule penalty function $S(d_m)$ that is bilinear with earliness penalty $e + e' < 1$ and lateness penalty $L + L'$. Since $e + e' < 1$, the simplified solution exists.

The user equilibrium solution is illustrated in Figure 3 in which all queuing occurs in the morning because the arrival curve in the morning determines not only the departure curve in the morning but also both curves in the evening. The user equilibrium cumulative arrival curve in the morning, $A_m(t)$, and evening, $A_e(t)$, must satisfy:

$$\dot{A}_m(t) = \begin{cases} \frac{\mu}{1-(e+e')} & \text{for early commuters} \\ \frac{\mu}{1+(L+L')} & \text{for late commuters} \end{cases} \tag{36}$$

$$\dot{A}_e(t) = \mu. \tag{37}$$

The bottleneck arrival curve in the evening must also start at time Δ after the morning departures, so $A_e(t) = D_m(t - \Delta)$.

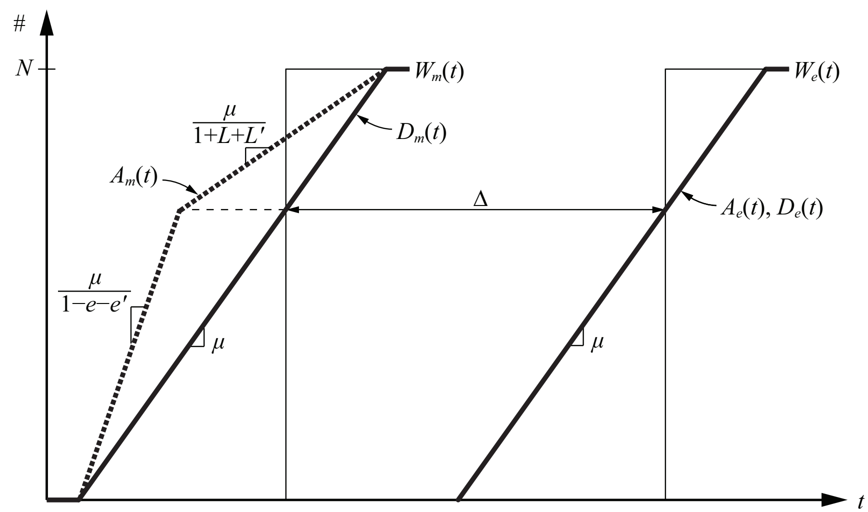

Fig. 3. User equilibrium for the combined morning and evening commutes with rigid work shifts of length Δ.

The resulting queues can be eliminated making sure that each commuter pays the equivalent of their queuing cost as a toll. This toll could be administered entirely within the morning when the user equilibrium queuing is experienced. However, the morning and evening commutes are intrinsically linked by the rigid work shift, so the toll for each individual may be split between the morning and evening in any proportion (e.g., half and half, or all in the evening), because each commuter chooses only the morning arrival time in response to the costs of the morning and evening commutes.

4.3. Fixed Wish Order with Cars and Transit

We finally consider a more general case of the combined morning and evening commute in which the wished bottleneck departure times in the morning and arrival times in the evening are in the same order but may be distributed over time. Although restrictive, this is not entirely unrealistic because commuters who want to start work earlier in the morning are more likely to get off work earlier in the evening. We consider Z-shaped wish curves with slope λ_m in the morning and λ_e in the evening. The bottleneck can serve cars and transit as described in Section 2.1, and the schedule penalty for each commuter is a separable function as described in Section 4.1. For simplicity, we present the equilibrium solution when the peaked demand is high enough that $\lambda_m > \mu_o$ and $\lambda_e > \mu_o'(1 + e')$. The transit capacity is always proportional to μ_o so that the total number of commuters traveling when both cars and transit are used is the same in the morning and evening. We also assume that commuters must use the same mode in the morning and evening; i.e., people who drive in the morning must take their car home in the evening, and people who ride transit in the morning must use transit to return home.

Just as for the single peak, the transit service is operated at a fixed headway, and the difference between the generalized cost of a one-way transit trip and a one-way free-flow car trip is $T_T = z_T - z_C$. Commuters are assumed to choose the times when they travel and whether to use car or transit in order to minimize the total generalized cost of their daily round trip including queuing and schedule penalties. Therefore, transit service will be competitive for all commuters that face a combined morning and evening queueing time of at least $2T_T$.

Figure 4 shows the cumulative wished, arrival, and departure curves for the morning and evening user equilibrium with cars and transit. The commuters that travel earliest experience less cost than a free-flow transit trip, so all travel by car. The slope of the equilibrium arrival curves in the morning and evening must satisfy (1) and (8), and the bottleneck serves commuters at rate μ in the morning and μ' in the evening when only cars are used. The queuing delay at B, after N_{AB} commuters have traveled, is $T_e = N_{AB}e/\mu$, and the queue at B′ is $T_e' = N_{A'B'}e'/\mu'(1 + e')$. We want to identify the points B and B′ at which transit becomes competitive. Since the number of early car commuters in the morning and evening must be the same, $N_{AB} = N_{A'B'}$. The value of N_{AB} is the one that makes the combined queueing time in the morning and evening add up to the cost of an uncongested transit trip, $T_e + T_e' = 2T_T$. By substitution, and solving for N_{AB}:

$$N_{AB} = \frac{2T_T}{\frac{e}{\mu} + \frac{e'}{\mu'(1+e')}}. \tag{38}$$

Note that T_e is not equal to T_e' unless $e/\mu = e'/\mu'(1 + e')$. This asymmetry is due to the fact that a commuter chooses to take transit based on the round trip cost, and this may mean taking a more costly commute in one rush in exchange for a less costly commute in another.

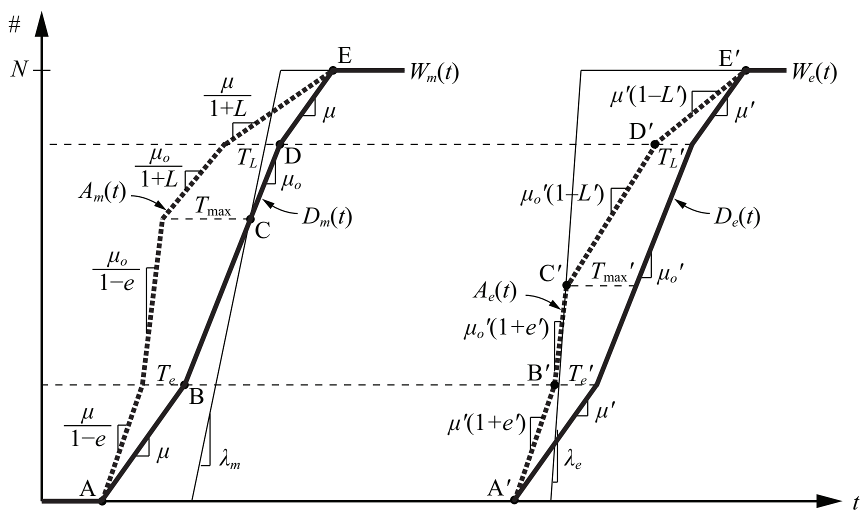

Fig. 4. User equilibrium for the combined morning and evening peaks with cars and transit.

Similar logic as used to determine N_{AB} can be used to identify how many commuters will travel late and use only their cars. The last N_{DE} commuters, starting at point D, travel only by car as the queueing time decreases and the transit is not competitive at the end of the rush. Following from the required slopes of the equilibrium arrival curves in (1) and (8), the queueing delay at D is $T_L = N_{DE}L/\mu$, and at D′ is $T_L' = N_{D'E'}L'/\mu'(1 - L')$. Just as for the early drivers, the number of late commuters traveling only by car in the morning and evening must match in order to conserve the flow of cars: $N_{DE} = N_{D'E'}$. The point D and D′ where transit is no longer competitive is the point where $T_L + T'_L = 2T_T$. This corresponds to:

$$N_{DE} = \frac{2T_T}{\frac{L}{\mu} + \frac{L'}{\mu'(1-L')}}. \tag{39}$$

Note again that T_L is only equal to T'_L if $L/\mu = L'/\mu'(1-L')$. Furthermore, note that the queueing delay at the start and end of transit use within morning and evening rushes need not be equal (i.e., $T_e \neq T_L$, $T'_e \neq T'_L$). In fact, the queue length at the start and end of transit use is only equivalent if $e\mu'(1+e')/e'\mu = L\mu'(1-L')/L'\mu$. As a result, the ratios of early and late commuters derived for the isolated morning commute in (7) and for the isolated evening commute in (11) no longer hold.

The remaining commuters that travel in the middle of the rush must also be the same in number in the morning and evening:

$$N_{BD} = N_{B'D'} = N - N_{AB} - N_{DE}. \tag{40}$$

These commuters travel by car and transit, and the bottleneck serves them at rate μ_o in the morning and μ'_o in the evening. The slope of the equilibrium arrival curves for these commuters must also satisfy (1) and (8). The number of early and late commuters in this middle period of the rush are determined by these slopes. Figure 5 shows how the delays relate to queueing position in the morning and evening rushes based on these arrival slopes. In the figure, the on time commuter in the morning is at a later position than the on time commuter in the evening. However, this is a consequence of the relative penalties for earliness and lateness that are chosen for the figure, and this result could be switched with another schedule penalty function.

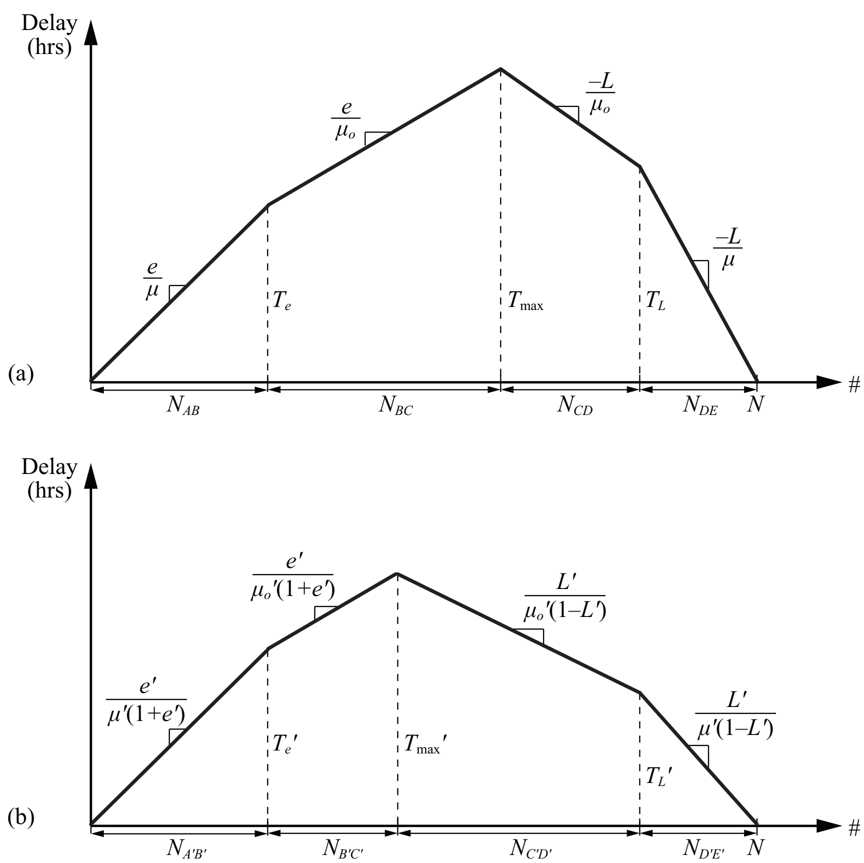

Fig. 5. Queueing delay in (a) the morning peak and (b) the evening peak for the combined user equilibrium with cars and transit.

It follows from the slopes of the arrival curves as illustrated in Figures 5a and b that the number of early commuters, N_{BC}, and the number of late commuters, N_{CD}, in the middle of the morning rush must satisfy the following:

$$T_{\max} = T_e + N_{BC}e/\mu_o = T_L + N_{CD}L/\mu_o \tag{41}$$

$$N = N_{AB} + N_{BC} + N_{CD} + N_{DE}. \tag{42}$$

The equilibrium values of N_{BC} and N_{BD} are the solution to this system of equations. Likewise, the number of early and late commuters in the middle of the evening rush, $N_{B'C'}$ and $N_{C'D'}$, must satisfy the following:

$$T'_{\max} = T'_e + N_{B'C'}e'/\mu'_o(1+e') = T'_L + N_{C'D'}L'/\mu'_o(1-L') \tag{43}$$

$$N = N_{A'B'} + N_{B'C'} + N_{C'D'} + N_{D'E'}. \tag{44}$$

Note that due to the asymmetry of the schedule penalties in the morning and evening, the number of early and late commuters in the middle period of the rush are not necessarily equal. The critical commuter who travels on time in the morning will be the same as the critical commuter traveling on time in the evening if $e/\mu = e'/\mu'(1+e')$ and $L/\mu = L'/\mu'(1-L')$.

We have shown the user equilibrium travel pattern for the combined morning and evening commutes with cars and transit by specifying the slopes of the arrival curves in each peak and the number of commuters early by car only, late by car only, and in the middle by car and transit. We now turn our attention to the number of commuters choosing to use transit in the combined commute. We compare this number with the number of transit users that would be expected in an isolated morning or evening commute if we had not considered the cost of the commutes together.

Proposition 2 (Transit Use in the Combined Commute)**.** *If commuters travel in combined morning and evening commute with common wished order, then there are at least as many transit riders in the combined user equilibrium as there are in the isolated morning and evening commutes together.*

Proof. The bottleneck is assumed to be fully utilized at capacity μ_o in the middle of the morning rush and μ'_o in the middle of the evening rush. The transit capacity, which is assumed to be a fixed proportion of this combined capacity, must be fully utilized. The number of transit riders is $N_T = N_{BD}\mu_T/\mu_o = N_{BD}\mu'_T/\mu'_o$, and since the capacity is not changing during this middle part of the rush, N_T is always proportional to N_{BD}. Therefore, a comparison of the values of N_{BD} for the isolated and combined commutes is sufficient to compare the number of transit users.

We compare the number of mid-rush commuters from the isolated morning and evening together as calculated with twice the number of mid-rush commuters calculated for one direction of the combined user equilibrium. The number of commuters for the combined equilibrium must be doubled because $N_{BD} = N_{B'D'}$. The following expression is equivalent to the statement of the proposition:

$$\left(N - \frac{T_T\mu}{e} - \frac{T_T\mu}{L}\right) + \left(N - \frac{T_T\mu'(1+e')}{e'} - \frac{T_T\mu'(1-L')}{L'}\right) \leq 2\left(N - \frac{2T_T\mu}{\frac{e}{\mu} + \frac{e'}{\mu'(1+e')}} - \frac{2T_T\mu}{\frac{L}{\mu} + \frac{L'}{\mu'(1-L')}}\right) \tag{45}$$

where the left side is the sum from the isolated peaks, from (5) and (13), and the right side is the twice the value of N_{BD} for the combined peak, from (40). We will verify that (45) is true, thereby proving the proposition.

By subtracting $2N$ from each side and dividing by T_T, (45) simplifies to:

$$\frac{\mu}{e} + \frac{\mu}{L} + \frac{\mu'(1+e')}{e'} + \frac{\mu'(1-L')}{L'} \geq \frac{4}{\frac{e}{\mu} + \frac{e'}{\mu'(1+e')}} + \frac{4}{\frac{L}{\mu} + \frac{L'}{\mu'(1-L')}}. \tag{46}$$

The inequality in (46) holds if we consider only the terms with e and e' or only the terms with L and L':

$$\frac{\mu}{e} + \frac{\mu'(1+e')}{e'} \geq \frac{4}{\frac{e}{\mu} + \frac{e'}{\mu'(1+e')}} \tag{47}$$

$$\frac{\mu}{L} + \frac{\mu'(1-L')}{L'} \geq \frac{4}{\frac{L}{\mu} + \frac{L'}{\mu'(1-L')}}. \tag{48}$$

Multiplying the first expression by $\frac{e}{\mu}\frac{e'}{\mu'(1+e')}(\frac{e}{\mu}+\frac{e'}{\mu'(1+e')})$ and the second expression by $\frac{L}{\mu}\frac{L'}{\mu'(1-L')}(\frac{L}{\mu}+\frac{L'}{\mu'(1-L')})$, then subtracting the right hand side from the left and simplifying, (47) and (48) become

$$\left(\frac{e}{\mu}-\frac{e'}{\mu'(1+e')}\right)^2 \geq 0 \tag{49}$$

$$\left(\frac{L}{\mu}-\frac{L'}{\mu'(1-L')}\right)^2 \geq 0. \tag{50}$$

These expressions are true for any value of e, e', L, and L', so this verifies the statement of the proposition. □

5. Conclusion

This paper has presented the user equilibrium for the morning commute with cars and transit with a capacity constraint in which commuters wish to depart a bottleneck to get to a destination on time. The reverse problem for the evening commute has also been presented in which commuters wish to leave their origin and arrive at a bottleneck on time. It has been shown that the user equilibrium arrival curve in the evening is not simply the reverse of the user equilibrium arrival curve in the morning. This asymmetry occurs because schedule deviation in the morning is the difference between the departure curve and the wished curve whereas schedule deviation in the evening is the difference between the arrival curve and the wished curve.

Despite the asymmetry, the morning and evening user equilibrium do share some commonalities. The maximum queuing delay experienced by the on time traveler in the middle of the rush shares the same relation to the earliness and lateness penalties in the morning and evening rush as shown by the similarity of (6) and (14). The provision of competitive public transit also provides a benefit by reducing the total delay and schedule penalty in both peaks. The benefit of transit is reduced as the combined car and transit capacity, μ_o, becomes more constrained. In both the morning and evening, the benefit of providing capacity-constrained transit service is $(1-\mu/\mu_o)/(1-\mu/\lambda)$ times the benefit of providing a transit service with no capacity constraint.

The system optimum in the morning and evening are symmetric, because without wasteful queuing delay, the arrival curve and departure are the same and the schedule penalty in the morning and evening can be accounted for as the difference between the departure curve and wished curve. This result is important, because it means that the system optimal tolls for the evening commute should not equal the cost of user equilibrium queueing as is commonly known for the morning commute.

The results of the analysis have also been extended to consider the combined morning and evening commutes. In the case that all commuters share identical wished times in the morning and evening and the schedule penalty is a function of the morning departure and evening arrival times with the properties described in Proposition 1, a user equilibrium has been shown to exist in which commuters travel in the same order in both peaks. The results are used to show the solutions for two special cases of interest: (i) the morning and evening commutes are independent, so each peak can be solved in isolation, and (ii) workers must work a rigid shift of length Δ but may choose when to start and end, so the two peaks collapse into a single morning commute problem.

Finally, the paper presents the user equilibrium for a special case of the combined commute with cars and transit in which commuters have the same wished order in the morning and evening. When commuters are constrained to use the same mode in the morning and evening, they make their travel decisions based on the combined cost that they will experience in the morning and evening peaks. The queueing delay when transit is first used may not equal the queuing delay when transit is last used, because commuters make their travel decisions in the face of the combined cost of the morning and evening commutes. One insight is that when the combined morning and evening peaks are considered transit ridership is at least as great as the sum of morning and evening riders that would be estimated by considering the morning and evening peaks in isolation.

The results of this paper highlight the differences between the morning and evening commute, and the effect of considering the equilibrium with both peaks together. The different characteristics of the morning and evening peaks make it necessary to treat them explicitly, because the equilibrium in the evening is not generally the same equilibrium in a morning commute. Furthermore, insights from the consideration of transit service and transit capacity are useful for planning transit services and understanding the factors that affect mode choice and the temporal distribution of demand.

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